DEVELOPMENT OF A SOLAR ABSORPTION LINE WITH A SPLIT UPPER LEVEL IN A MAGNETIC FIELD

V. N. Obridko

Institute of Terrestrial Magnetism, Ionosphere and Radiowave Propagation,
Academy of Sciences of the USSR

The transfer equation for a spectral line with a split upper level is solved formally. The explicit solution is then explored by the method of Chandrasekhar. Simplified expressions are given for the first approximation to the solution which is then compared with a solution obtained by Unno. Discrepancy between the two is quite large at the center of the line but falls off toward the wings.

The solar absorption line FeI λ 5250.218 Å due to the $a^5D_0 - z^7D_1$ transition is the one most frequently used at the present time in studies of the solar magnetic fields. In many cases, and especially in investigations involving the determination of the magnitude and direction of the magnetic field, it is important to have a complete knowledge of all the polarization parameters for all points of the line profile. Unno [1] has obtained a solution of the transfer equation for a line formed as a result of true absorption. The solution is used in most papers concerned with measurements of the solar magnetic field vector [2-5]. Unfortunately, scattering plays an important part in the formation of the λ 5250 Å line. This is evident from the fact that the line profile varies only slightly between the center and the wings. Stepanov [6-7] has obtained a solution of the transfer equations with allowance for scattering, but had to introduce a number of important simplifying assumptions. Rachkovskii [8] obtained an exact solution with allowance for scattering for a line with a split lower level ($j_l = 1$, $j_u = 0$). An example of such a line in the solar spectrum is the line FeI λ 6302.5 Å due to the $z^5P_1 - g^5D_0$ transition. Rachkovskii showed that discrepancies with Stepanov's theory are quite substantial at the center of the line, but fall off rapidly toward the wings. The problem of the formation of the λ 5250 Å line, which is different in so far as the upper layer is split ($j_l = 0$, $j_u = 1$), has remained unresolved. Below we report a solution of

the transfer equation for a line of this kind. The solution was obtained by the method of Chandrasekhar [9] on the assumption of azimuthal symmetry. Magnetic rotation in the line was not taken into account.

The transfer equations for polarized radiation in the presence of a magnetic field may be written in the form

$$\mu \frac{dI}{d\tau} = (1 + \eta_I)I + \eta_Q Q + \eta_V V - S_I,$$

$$\mu \frac{dQ}{d\tau} = \eta_Q I + (1 + \eta_I)Q - S_Q, \qquad (1)$$

$$\mu \frac{dV}{d\tau} = \eta_V I + (1 + \eta_I)V - S_V.$$

where I, Q, V are the usual Stokes parameters for polarized radiation [1, 9]:

$$\eta_I = \frac{\eta_p}{2}\sin^2\psi + \frac{\eta_l + \eta_r}{4}(1 + \cos^2\psi),$$

$$\eta_Q = \left(\frac{\eta_p}{2} - \frac{\eta_l + \eta_r}{4}\right)\sin^2\psi, \qquad (2)$$

$$\eta_V = \frac{-\eta_l + \eta_r}{2}\cos\psi.$$

ψ is the angle between the magnetic field and the line of sight and η_p, η_l, η_r are the absorption coefficients in the π_-, σ_+, σ_- components respectively. The three source functions S_I, S_Q, and S_V determine the different contributions of scattering and true absorption to each equation in (1).

To begin with let us write down the formal solution of (1) for an arbitrary source function:

$$I = -\frac{1}{2\mu}\left\{e^{\lambda_2\tau}\int\left(S_I + \frac{\eta_Q}{\sqrt{\eta_Q^2+\eta_V^2}}S_Q + \frac{\eta_V}{\sqrt{\eta_Q^2+\eta_V^2}}S_V\right)e^{-\lambda_2\tau}d\tau + e^{\lambda_3\tau}\int\left(S_I - \frac{\eta_Q}{\sqrt{\eta_Q^2+\eta_V^2}}S_Q - \frac{\eta_V}{\sqrt{\eta_Q^2+\eta_V^2}}S_V\right)e^{-\lambda_3\tau}d\tau\right\},$$

$$Q = -\frac{1}{\mu}e^{\lambda_1\tau}\int\eta_V\frac{S_Q\eta_V - S_V\eta_Q}{\eta_Q^2+\eta_V^2}e^{-\lambda_1\tau}d\tau - \frac{1}{2\mu}\frac{\eta_Q}{\sqrt{\eta_Q^2+\eta_V^2}}\left\{e^{\lambda_2\tau}\int\left(S_I + \frac{\eta_Q}{\sqrt{\eta_Q^2+\eta_V^2}}S_Q + \frac{\eta_V}{\sqrt{\eta_Q^2+\eta_V^2}}S_V\right)\right.$$

$$\left. \times e^{-\lambda_2\tau}d\tau - e^{\lambda_3\tau}\int\left(S_I - \frac{\eta_Q}{\sqrt{\eta_Q^2+\eta_V^2}}S_Q - \frac{\eta_V}{\sqrt{\eta_Q^2+\eta_V^2}}S_V\right)e^{-\lambda_3\tau}d\tau\right\},$$

$$(3)$$

$$V = \frac{1}{\mu}e^{\lambda_1\tau}\int\eta_Q\frac{S_Q\eta_V - S_V\eta_Q}{\eta_Q^2+\eta_V^2}e^{-\lambda_1\tau}d\tau - \frac{1}{2\mu}\frac{\eta_V}{\sqrt{\eta_Q^2+\eta_V^2}}\left\{e^{\lambda_2\tau}\int\left(S_I + \frac{\eta_Q}{\sqrt{\eta_Q^2+\eta_V^2}}S_Q + \frac{\eta_V}{\sqrt{\eta_Q^2+\eta_V^2}}S_V\right)\right.$$

$$\left. \times e^{-\lambda_2\tau}d\tau - e^{\lambda_3\tau}\int\left(S_I - \frac{\eta_Q}{\sqrt{\eta_Q^2+\eta_V^2}}S_Q - \frac{\eta_V}{\sqrt{\eta_Q^2+\eta_V^2}}S_V\right)e^{-\lambda_3\tau}d\tau\right\}.$$

where

$$\lambda_1 = -\frac{1+\eta_I}{\mu}, \qquad \lambda_{2,3} = \frac{1+\eta_I \pm \sqrt{\eta_Q^2+\eta_V^2}}{\mu}.$$

In the case of true absorption

$$S_I = (1+\eta_I)B_\nu,$$
$$S_Q = \eta_Q B_\nu, \qquad S_V = \eta_V B_\nu$$

and for $B_\nu = B_0(1 + \beta_T)$ we have Unno's solution

$$I_u = 1 + \beta\mu\frac{1+\eta_I}{(1+\eta_I)^2 - \eta_Q^2 - \eta_V^2},$$

$$Q_u = -\beta\mu\frac{\eta_Q}{(1+\eta_I)^2 - \eta_Q^2 - \eta_V^2}, \qquad (4)$$

$$V_u = -\beta\mu\frac{\eta_V}{(1+\eta_I)^2 - \eta_Q^2 - \eta_V^2}.$$

When there is scattering, the source function depends substantially on I, Q, and V, and the particular form of this dependence is determined by the type of splitting. For the 5250.22 Å line, the source function can be obtained with the aid of the scattering matrix which we introduced in [10]:

$$S_I = \frac{3}{2}(1-\varepsilon)\int\left[\frac{\eta_l+\eta_r}{8}(1+\cos^2\psi)(1+\cos^2\psi') + \frac{\eta_p}{2}\sin^2\psi\sin^2\psi'\right]I\frac{d\omega'}{4\pi} + \frac{3}{2}(1-\varepsilon)\int\left[\frac{\eta_p}{2}\sin^2\psi\sin^2\psi'\right.$$

$$\left. -\frac{\eta_l+\eta_r}{8}(1+\cos^2\psi)\sin^2\psi'\right]Q\frac{d\omega'}{4\pi} + \frac{3}{2}(1-\varepsilon)\int\left[\frac{-\eta_l+\eta_r}{4}(1+\cos^2\psi)\cos\psi'\right]V\frac{d\omega'}{4\pi} + (1+\varepsilon\eta_I)B_\nu,$$

$$S_Q = \frac{3}{2}(1-\varepsilon)\int\left[\frac{\eta_p}{2}\sin^2\psi\sin\psi' - \frac{\eta_l+\eta_r}{8}(1+\cos^2\psi')\sin^2\psi\right]I\frac{d\omega'}{4\pi} \qquad (5)$$

$$+\frac{3}{2}(1-\varepsilon)\int\left[\frac{\eta_p}{2}\sin^2\psi\sin^2\psi' + \frac{\eta_l+\eta_r}{8}\sin^2\psi\sin^2\psi'\right]Q\frac{d\omega'}{4\pi} - \frac{3}{2}(1-\varepsilon)\int\frac{-\eta_l+\eta_r}{4}\sin^2\psi\cos\psi'V\frac{d\omega'}{4\pi} + \varepsilon\eta_Q B_\nu,$$

$$S_V = \frac{3}{2}(1-\varepsilon)\int\left[\frac{-\eta_l+\eta_r}{4}(1+\cos^2\psi')\cos\psi I - \frac{-\eta_l+\eta_r}{4}\sin^2\psi'\cos\psi Q + \frac{\eta_l+\eta_r}{2}\cos\psi'\cos\psi V\right]\frac{d\omega'}{4\pi} + \varepsilon\eta_V B_\nu.$$

The source functions in (1) may be obtained by solving (1) by the method of Chandrasekhar [9] which, however, is applicable only to azimuthally symmetric problems in which $\mu = \cos\psi$, and (1) and (5) can be written for $\varepsilon = 0$ as follows:

$$\mu_i\frac{dI_i}{d\tau} = (1+\eta_{Ii})I_i + \eta_{Qi}Q_i + \eta_{Vi}V_i - \frac{3}{4}\frac{\eta_l+\eta_r}{8}(1+\mu_i^2)\sum_i,$$

$$I_i(1+\mu_i^2)\alpha_i - \frac{3}{4}\frac{\eta_p}{2}(1-\mu_i^2)\sum_i I_i(1-\mu_i^2)\alpha_i + \frac{3}{4}\frac{\eta_l+\eta_r}{8}(1+\mu_i^2)\sum_i Q_i(1-\mu_i^2)\alpha_i$$

$$-\frac{3}{4}\frac{\eta_p}{2}(1-\mu_i^2)\sum_i Q_i(1-\mu_i^2)\alpha_i - \frac{3}{4}\frac{-\eta_l+\eta_r}{4}(1+\mu_i^2)\sum_i V_i\mu_i\alpha_i - B_\nu$$

$$\mu_i \frac{dQ_i}{d\tau} = \eta_{Qi} I_i + (1 + \eta_{Ii}) Q_i + \frac{3}{4} \frac{\eta_l + \eta_r}{8} (1 - \mu_i^2) \sum_i I_i (1 + \mu_i^2) \alpha_i - \frac{3}{4} \frac{\eta_p}{2} (1 - \mu_i^2) \sum_i I_i (1 - \mu_i^2) \alpha_i$$

$$- \frac{3}{4} \frac{\eta_l + \eta_r}{8} (1 - \mu_i^2) \sum_i Q_i (1 - \mu_i^2) \alpha_i - \frac{3}{4} \frac{\eta_p}{2} (1 - \mu_i^2) \sum_i Q_i (1 - \mu_i^2) \alpha_i + \frac{3}{4} \frac{-\eta_l + \eta_r}{4} (1 - \mu_i^2) \sum_i V_i \mu_i \alpha_i,$$

$$\mu_i \frac{dV_i}{d\tau} = \eta_{Vi} I_i + (1 + \eta_{Ii}) V_i - \frac{3}{4} \frac{-\eta_l + \eta_r}{4} \mu_i \sum_i I_i (1 + \mu_i^2) \alpha_i$$

$$+ \frac{3}{4} \frac{-\eta_l + \eta_r}{4} \mu_i \sum_i Q_i (1 - \mu_i^2) \alpha_i - \frac{3}{4} \frac{\eta_l + \eta_r}{2} \mu_i \sum_i V_i \mu_i \alpha_i. \qquad (6)$$

We have thus replaced a system of three integro-differential equations by a system of 6n linear first-order differential equations where $\mu_i = -\mu_{-i}$ are the zeros of the Legendre polynomials $P_{2n}(\mu)$, and $\alpha_i = \alpha_{-i}$ are the corresponding Gaussian weights.

The solution of this system is

$$\frac{I_i}{B} = \sum_{\alpha=1}^{3n} C_{I\alpha} e^{-k_\alpha \tau} + 1 + \beta\tau$$

$$+ \beta\mu_i \frac{1 + \eta_{Ii}}{(1 + \eta_{Ii})^2 - \eta_{Qi}^2 - \eta_{Vi}^2}, \qquad (7)$$

$$\frac{Q_i}{B_0} = \sum_{\alpha=1} C_{Q\alpha} e^{-k_\alpha \tau} - \beta\mu_i \frac{\eta_{Qi}}{(1 + \eta_{Ii})^2 - \eta_{Qi}^2 - \eta_{Vi}^2},$$

$$\frac{V_i}{B_0} = \sum_{\alpha=1}^{3n} C_{V\alpha} e^{-k_\alpha \tau} - \beta\mu_i \frac{\eta_{Vi}}{(1 + \eta_{Ii})^2 - \eta_{Qi}^2 - \eta_{Vi}^2}.$$

and since there are no sources on the boundary, we have

$$\sum_{\alpha=1}^{3n} C_{I\alpha} = -1 - \beta\mu_i \frac{1 + \eta_{Ii}}{(1 + \eta_{Ii})^2 - \eta_{Qi}^2 - \eta_{Vi}^2},$$

$$\sum_{\alpha=1}^{3n} C_{Q\alpha} = \beta\mu_i \frac{\eta_{Qi}}{(1 + \eta_{Ii})^2 - \eta_{Qi}^2 - \eta_{Vi}^2}, \qquad (8)$$

$$\sum_{\alpha=1}^{3n} C_{V\alpha} = \beta\mu_i \frac{\eta_{Vi}}{(1 + \eta_{Ii})^2 - \eta_{Qi}^2 - \eta_{Vi}^2}.$$

We have not succeeded in obtaining an explicit expression for the characteristic roots and proper values of the matrix of the system given by (6) for $n > 1$. The characteristic roots and the proper values of the matrix were therefore calculated by a modification of Krylov's method [11].

The solution given by (7) may be used to find the source functions, which can then be substituted into the formal solution given by (3). The final solution of (1) will then be

$$I(0, \mu) = \sum_{\alpha=1}^{3n} \frac{(1 + k_\alpha\mu + \eta_I) P_{I\alpha} - \eta_Q P_{Q\alpha} - \eta_V P_{V\alpha}}{(1 + k_\alpha\mu + \eta_I)^2 - \eta_Q^2 - \eta_V^2} + I_u,$$

$$Q(0, \mu) = \sum_{\alpha=1}^{3n} \eta_Q \frac{-(\eta_Q^2 + \eta_V^2) P_{I\alpha} + (1 + k_\alpha\mu + \eta_I)(\eta_Q P_{Q\alpha} + \eta_V P_{V\alpha})}{[(1 + k_\alpha\mu + \eta_I)^2 - \eta_Q^2 - \eta_V^2](\eta_Q^2 + \eta_V^2)} + \frac{\eta_V}{\eta_Q^2 + \eta_V^2} \sum_{\alpha=1}^{3n} \frac{\eta_V P_{Q\alpha} - \eta_Q P_{V\alpha}}{1 + k_\alpha\mu + \eta_I} + Q_U,$$

$$V(0, \mu) = \sum_{\alpha=1}^{3n} \eta_V \frac{-(\eta_Q^2 + \eta_V^2) P_{I\alpha} + (1 + k_\alpha\mu + \eta_I)(\eta_Q P_{Q\alpha} + \eta_V P_{V\alpha})}{[(1 + k_\alpha\mu + \eta_I)^2 - \eta_Q^2 - \eta_V^2](\eta_Q^2 + \eta_V^2)} - \frac{\eta_Q}{\eta_Q^2 + \eta_V^2} \sum_{\alpha=1}^{3n} \frac{\eta_V P_{Q\alpha} - \eta_V P_{V\alpha}}{1 + k_\alpha\mu + \eta_I} + V_U.$$

The solution for a line with any type of splitting can be written in this form. The particular type of splitting will affect the quantities k_α, $P_{I\alpha}$, $P_{V\alpha}$ and $P_{Q\alpha}$ · $P_{I\alpha}$, $P_{Q\alpha}$ and $P_{V\alpha}$. The last three are found by substituting (7) in the expression for the source functions. The result is

$$S_I = \sum_\alpha P_{I\alpha} e^{-k_\alpha \tau} + (1 + \eta_I) B_\nu,$$

$$S_Q = \sum_\alpha P_{Q\alpha} e^{-k_\alpha \tau} + \eta_Q B_\nu, \qquad (10)$$

$$S_V = \sum_\alpha P_{V\alpha} e^{-k_\alpha \tau} + \eta_V B_\nu.$$

The solution assumes a simpler form for n = 1, in which the characteristic equation is a cubic in $\mu_1^2 k^2$:

$$\mu_1^6 k^6 - (1+\eta_{I1})(3+\eta_{I1})\mu_1^4 k^4 + \left\{ -\eta_{Q1}\eta_{V1}^2 + \left[(1+2\eta_{QQ1})(1+2\eta_{VV1}) - \frac{\eta_{V1}^2}{4} \right](1+\eta_{I1})^2 \right.$$

$$- \eta_{V1}^2(1+2\eta_{QQ1}) - \eta_{Q1}^2(1+2\eta_{VV1}) + (1+\eta_{I1})^2(2+\eta_{I1}) \bigg\} \mu_1^2 k^2$$

$$- (1+\eta_{I1})[(1+\eta_{I1})^2 - \eta_{Q1}^2 - \eta_{V1}^2] \left[(1+2\eta_{QQ1})(1+2\eta_{VV1}) - \frac{\eta_{V1}^2}{4} \right].$$

where

$$\eta_{QQ} = \left(\frac{\eta_p}{4} + \frac{\eta_e + \eta_r}{16} \right) \sin^2 \psi,$$

$$\eta_{VV} = \frac{\sqrt{3}}{8}(\eta_e + \eta_r)\cos\psi.$$

Let

$$\theta = [(1+2\eta_{VV1})(1+\eta_{I1}) - \mu_1^2 k_\alpha^2][(1+2\eta_{QQ1})(1+\eta_{I1})$$

$$- \mu_1^2 k_\alpha^2] - \frac{\eta_{V1}^2(1+\eta_{I1})^2}{4}$$

$$\theta_1 = \mu_1 k_\alpha \eta_{Q1}[(1+2\eta_{QQ1})(1+\eta_{I1}) - \mu_1^2 k_\alpha^2]$$

$$- \frac{\eta_{V1}^2(1+\eta_{I1})\mu_1 k_\alpha}{2}$$

$$\theta_2 = \mu_1 k_\alpha \eta_{V1}[(1+2\eta_{VV1})(1+\eta_{I1}) - \mu_1^2 k_\alpha^2]$$

$$- \frac{\eta_{V1}\eta_{Q1}(1+\eta_{I1})\mu_1 k_\alpha}{2},$$

$$x = \frac{2\mu_1 k_\alpha}{1+\mu_1 k_\alpha} L_\alpha, \quad z = -\frac{\theta_1}{\theta}\frac{2}{1+\mu_1 k_\alpha} L_\alpha,$$

$$w = -\frac{\theta_2}{\theta}\frac{2}{1+\mu_1 k_\alpha} L_\alpha,$$

$$t = -\frac{\eta_{V1}}{2\mu_1 k_\alpha} w - \frac{1+2\eta_{VV1}}{2\mu_1 k_\alpha} z$$

$$u = -\frac{1+2\eta_{QQ1}}{\mu_1 k_\alpha} w - \frac{\eta_{V1}}{2\mu_1 k_\alpha} z.$$

In that case

$$P_{I\alpha} = \frac{\eta_I}{2} x + \frac{\eta_Q}{2} z + \frac{\eta_V}{2} w,$$

$$P_{Q\alpha} = \frac{\eta_Q}{2} x + \eta_{QQ} z - \frac{\eta_V}{4} w,$$

$$P_{V\alpha} = \frac{\eta_V}{2} x - \frac{\eta_V}{4} z + \eta_{VV} w,$$

where L_α is found from the condition

$$\sum_{\alpha=1}^{3} L_\alpha = -1 + \beta\mu_1 \frac{1+\eta_{I1}}{(1+\eta_{I1})^2 - \eta_{Q1}^2 - \eta_{V1}^2},$$

$$\frac{1}{2}\sum_{\alpha=1}^{3}(z-t) = -\beta\mu_1 \frac{\eta_{Q1}}{(1+\eta_{I1})^2 - \eta_{Q1}^2 - \eta_{V1}^2},$$

$$\frac{1}{2}\sum_{\alpha=1}^{3}(u-w) = -\beta\mu_1 \frac{\eta_{V1}}{(1+\eta_{I1})^2 - \eta_{Q1}^2 - \eta_{V1}^2}.$$

Figure 1 shows a plot of the profile of the 5250.22 Å line. It was assumed that the absorption coefficient within the line is of the Doppler kind:

$$\eta_p = \eta_0 e^{-v^2}, \quad \eta_l = \eta_0 e^{-(v+v_H)^2}, \quad \eta_r = \eta_0 e^{-(v-v_H)^2}$$

$$v = \frac{\Delta\lambda}{\Delta\lambda_D}, \quad v_H = \frac{\Delta\lambda_H}{\Delta\lambda_D}, \quad \eta_0 = 9, \quad v_H = 1.0.$$

This was compared with Unno's solution calculated from Eq. (4) for $\mu = 1$. Figure 2 shows a plot of

$$\Delta = \frac{Q_{(9)} - Q_{(4)}}{Q_{(9)}}.$$

It is evident from Fig. 2 that near the center of the line, Unno's solution departs substantially from the solution obtained here, but in the wings the difference is smaller. The similarity between Unno's profile and our profile persists only near the center of the disc. As one approaches the limb, Unno's solution tends to zero, whereas our solution varies little. It is important to note that in photoelectric measurements of the total vector, the exit
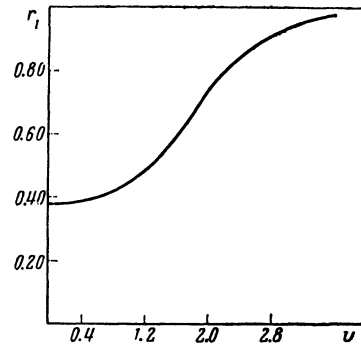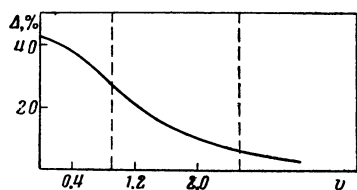

Fig.

Fig. 2

slit of the magnetograph intercepts radiation from a relatively broad section of the line wing (the size of the slit used in [2] is indicated by the broken lines in Fig. 2). It follows that as a result of integration over a wide slit, there is a large contribution due to those parts of the wing for which Δ is small. It is quite easy to show that when $v_l = 1.0$, the signal calculated from (9) will be larger than that obtained from Unno's solution by only 10%.

Therefore, if we take into account the weak dependence of (9) on μ, we can interpret observations of the total magnetic field at all points on the solar disc by using Unno's solution with $\mu = 1$ as was done in [2]. The errors introduced by this procedure will evidently be small. It is important to note, however, that so far the comparison has been carried out for a single magnitude of the splitting. Moreover, azimuthal asymmetry has not been taken into account in the solution. We intend to perform more detailed calculations. On the other hand it has been suggested that the magnetic field parameters can be deduced by comparing theoretical and observed profiles [1, 4, 5]. In such cases, the calculations should be carried out on the basis of the solution given by (9) above if the 5250.22 Å line

is employed or only the distant wings of the line should be considered.

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